The North Carolina Association Of Advanced Placement® Mathematics Teachers Newsletter

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*NOTE: David Royster and Norma Royster moved to Kentucky this past summer. We will miss them in NC and we appreciate their service to the NCA^2PMT Board. Any NC member wishing to be considered for nomination to the Board should contact any of the present board members listed above.

Notes from the President's Desk

It is hard to realize, but, May is less than three months away. Hopefully you will complete the AP curriculum soon and can begin an intense review for the AP Exams. I suggest that you extensively exploit the resources on AP Central and order the 2008 Multiple Choice Exam that has recently been released. The free response questions for the last ten years are on line with the rubrics used for scoring.

This is my last letter. Martha Ray will become your President in June when the NCAAPMT Board meets. Thank you for allowing me to serve you for the last two years. It has been a joy.

Gloria Nan Dupree, President
C. D. Owen High School, Black Mountain, NC 28711

Notes From The Secretary's Desk

Life in Western NC has been interesting at my house. Our snows have been pretty as well as a little disruptive to school schedules. AP teachers always worry about the loss of class time prior to AP exams. In WNC, the extra stress AP teachers have due to weather cancellations is a part of the workload they would rather not have. Since our last newsletter, I have been doing AP workshops and am looking forward to a few Saturday test preparation days with some VA students. My work with College Board has included a great opportunity to work with a mentor (Virginia from Connecticut) to improve our one-day workshop . This experience of having someone helping you see things from multiple perspectives has been fun – and I have a new AP friend. We will both be at the AP Reading this summer in Kansas City. I will try and talk her into writing an article for us in the Summer Newsletter.

If anyone is interested in taking over for me as editor of this newsletter, please let me know. I think it is time for new ideas and for someone who might like to design a new format. Anyone going to the AP Reading, please contact me ASAP - we need writers for Summer newsletter. I would like more NC writers to step forward.

On the personal side, I continue to search for full time work but love having more time at home and flexibility in my schedule. Finances are tight with both my sons in college. My oldest has returned to NCSU, hoping for another degree. The youngest is at ETSU playing bluegrass- both banjo and guitar and we are praying he ends up with a degree!

Deb Britt, Mars Hill, NC, dgb531@aol.com

Please remember to renew your membership to receive the two yearly newsletters. You can send your $5.00 check, payable to NCA^{2}PMT, to Jeff Lucia, 718 Lansdowne Road, Charlotte, NC 28270. Email address is jeff.lucia@providenceday.org.

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The Importance of Multiple Representations
Dan Kennedy - The Baylor School, Chattanooga, TN

One of the most valuable problem-solving talents a student can bring to higher mathematics courses from high school courses in algebra and geometry is the ability to represent a problem in multiple ways: algebraically, graphically, numerically, and verbally. The ability to switch representations is often the key to accessing paths to the solution that might otherwise remain hidden from view. Some of the most important discoveries in the history of mathematics have actually resulted from switching representations, for example:

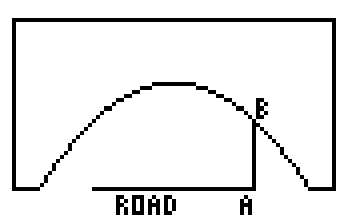
- Pythagoras knew that the hypotenuse of an isosceles right triangle with legs of unit length would have a length whose square would equal 2. This was problematic, as he knew of no such number. For Pythagoras, it was an unfortunate shortcoming of the process of measurement. For other mathematicians who looked at the problem numerically, it provided the opportunity to discover irrational numbers.
- One of the most famous geometric problems of antiquity was to trisect an arbitrary angle using only a compass and an unmarked straightedge. Geometrically, it remained an unsolved problem for two thousand years. We now know that it is impossible to trisect certain angles with a compass and straightedge, thanks to a 19th-century proof that re-cast the problem in terms of solving 3rd-degree algebraic equations.
- Before calculus, scientists were unable to use algebra to describe motion as a function of time, essentially because anything at a moment in time is apparently motionless. T hen a breakthrough in geometry (analytic geometry -- what we call "graphing") allowed mathematicians to envision a method for solving motion problems in terms of slopes and areas. Newton and Leibniz, working independently, were then able to revolutionize algebra so that it could model that geometric method, and that was the beginning of calculus. Indeed, that significant switch of representations ultimately kicked off the Age of Enlightenment.

The following are a few problems from high school mathematics that show the usefulness of switching representations.

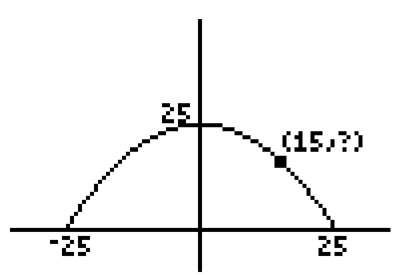
<u>From Verbal to Graphical to Algebraic: The Parabolic Underpass</u>

A bridge is supported by a parabolic arch that spans 50 feet, with a 30-foot wide road passing through the center. The arch is 25 feet tall at its highest point. What is the clearance at the edge of the road? (This is an important consideration for truckers.)

Solution: The first step is to draw the parabolic arch, thus representing the problem graphically. The problem is then geometric: find the length of segment AB. Unfortunately; our geometric tools are not well-suited to solving the problem, so we switch representations to algebra.



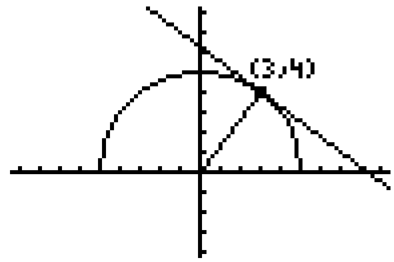
The parabola can be coordinatized so it has vertex (0, 25). Such a parabola has equation $y = 25 - ax^2$ for some *a*, and the *x*-intercepts of $\pm$ 25 enable us to determine that *a* = 1/25. Now we can easily use the algebraic equation $y = 25 - x^2/25$ to find the *y*-coordinate at *x* = 15, which is 16.
The clearance at the edge of the road is 16 feet.



From Algebraic to Geometric to Algebraic: The Tangent to the Curve

Find an equation of the line tangent to the graph of $y = \sqrt{25 - x^2}$ at the point (3, 4).

Solution: This looks in its algebraic representation like an advanced problem that would call for an understanding of differential calculus. If we represent the problem graphically, however, we see the solution as the familiar geometric construction of a line tangent to a semicircle at a point.



The line must be perpendicular to the radius at the point (3, 4). Switching representations back to algebra, we compute the slope of the radius to be 4/3 and conclude that the slope of the tangent line is –3/4. The point-slope equation of the line is $y - 4 = -0.75(x - 3)$.

From Numerical to Graphical to Algebraic: Pennsylvania Population

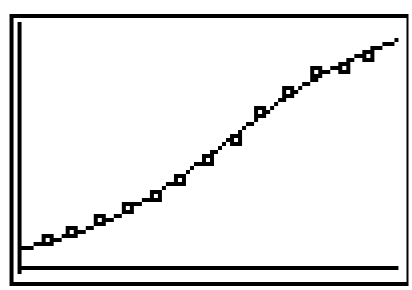
The table below shows the population of Pennsylvania in each 10-year census between 1830 and 1950. Use the data to predict the Pennsylvania population in the year 2000.

Years since 1820	Population in thousands
10	1348
20	1724
30	2312
40	2906
50	3522
60	4283
70	5258
80	6302
90	7665
100	8720
110	9631
120	9900
130	10498

Source: Bureau of the Census, U.S. Chamber of Commerce

Solution: This sort of "data-driven" problem was once rare in high school textbooks but has become fairly common since the advent of graphing calculators. In fact, we have chosen one that illustrates logistic growth just to give it a fresh face.

The solution begins, as is usually the case with such problems, by representing the numerical data graphically in a scatter plot (our first representation shift). The shape of the plot suggests that the growth, exponential at first, started bending the other way sometime soon after 1890. This change in curvature is the hallmark of logistic growth. We use logistic regression to find a curve that fits the data, thus changing representations again: from graphical to algebraic.



[0, 140] by [-200, 12000]

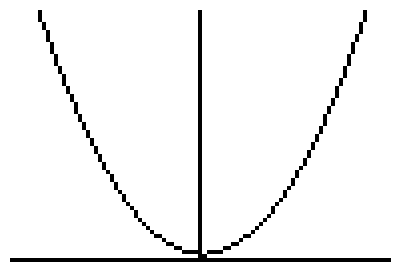
The regression equation (which might be slightly different on your calculator, as different calculators use different logistic models) is $y = \frac{12655.179}{1+12.871e^{-0.0326x}}$. Plugging x = 180 into this equation, we get a predicted population of 12,209,870 for the year 2000. This is remarkably close to the 2000 census number of 12,281,054.

Important real-world disclaimer: Predicting the future on the basis of past data is always risky business, especially with statistics like population that are dependent on so many sociological variables. Extrapolating 50 years beyond a 120-year data set is rarely as successful as in the problem above. Imagine, for example, what might have happened if we had been using data gathered between 1830 and 1950 to predict the American sales of women's hats in the year 2000.

From Algebraic to Numerical: The Ready-for Prime-Time Polynomial

The quadratic polynomial $y = x^2 - 41x + 41$ has an amazing mathematical property. What is it?

Solution In its algebraic form this polynomial appears to be no more amazing than any other, which is to say not amazing at all. If we switch representations to the graph we get a pretty mundane parabola, which is shown at the right in the window $[-47, 47]$ by $[0,1700]$.



So what's so amazing about it? You have to get numerical to find out. Use your calculator to make a table of the values you get when you evaluate this polynomial at positive integer values: 1, 2, 3, 4, etc. (The beginning of the table is shown to the right.) Scroll from X = 1 all the way down to X = 40. Do you see what the Y values have in common?

X	Y1	
1	41	
2	43	
3	47	
4	53	
5	61	
6	71	
7	83	

X=1

Number aficionados will probably recognize that the first several function values all appear to be prime numbers. In fact, they are *all* prime, at least until you get to X = 41 (for which $X^2 - X + 41$ should clearly not be prime). Since no function has ever been found that will generate prime numbers ad infinitum, it is a pleasant mathematical surprise to find such a simple polynomial that generates primes so spectacularly well. Leonhard Euler, who specialized in finding pleasant mathematical surprises, was also the first to appreciate this one.

Conclusion

The mathematics used to solve the four examples above is not especially profound. What gives the solutions their richness is the mathematical advantage the solver gains by switching representations in order to make full use of the mathematics that is available. Problem-solving is as much an art as it is a science, and if we are to teach the art to our students it is not enough to teach them the mathematics. That is why teachers at every level should continually expose their students to algebraic, graphical, numerical, and verbal representations of the concepts they are teaching.

- Use multiple representations whenever possible in classroom examples.
- Assign homework problems that employ a variety of representations, augmenting textbook exercises if necessary.

- Use classroom time for student problem-solving, always seeking out and discussing alternate paths that some students might find to the solution.
- Encourage the use of graphing calculators (which are themselves technological tools for switching representations) when solving problems.

Avoid giving students the impression that there is only one way to solve a problem, even when you believe there might be. They might surprise you!

Five "Pledged" Problems - Fall, 2009
Ben Klein - Davidson College, Davidson, NC

Note: Expectations are that students would have a TI-89 to work these problems, and some kind of CAS is critical for some of them.

INSTRUCTIONS: Do each part of these pledged problem sets. Organize your work and write up your answers neatly; inordinately messy papers will not be accepted. In working on these problems, you may use your textbook, your class notes and your calculator. If you want to use any other reference or computational aid, you must check with Dr. Klein first. You may not discuss the problems with anyone other than Dr. Klein, who will be happy to clarify the statements of the parts for you. **BE SURE TO PLEDGE YOUR PAPER BEFORE YOU SUBMIT IT; UNPLEDGED SOLUTIONS WILL NOT BE ACCEPTED.**

Pledged Problem Set #1

FACTS: (1) $\lim_{x\to 0}\left[\frac{\sin(x)}{x}\right]=1$ and (2) neither of the one-sided limits $\lim_{x\to +0^+}\left[\sin\left(\frac{1}{x}\right)\right]$ and $\lim_{x\to +0^-}\left[\sin\left(\frac{1}{x}\right)\right]$ exist. We proved (1) on 3 September and proved the two-sided version of (2) on 1 September.

INSTRUCTIONS: Evaluate the following limits, giving, in each case, an algebraic or trigonometric justification for your answer. If you conclude that a given limit does not exist, be sure to say so and explain why. You can (should) use your TI-89 to check your answers.

1) Find $\lim_{x\to 0}\left[\frac{\sin^2(x)}{x}\right]$. HINT: $\sin^2(x)=\sin(x)\cdot\sin(x)$.

2) Find $\lim_{x\to 0}\left[\frac{\sin(x^2)}{x}\right]$. HINT: $\frac{1}{x}=x\frac{1}{x^2}$. Let $t=x^2$ in part of the resulting expression.

3) Find the one sided limits: $\lim_{x\to 0^+}\left[\frac{\sin(x)}{x^2}\right]$ and $\lim_{x\to 0^-}\left[\frac{\sin(x)}{x^2}\right]$.

4) Find the one sided limit: $\lim_{x\to 0^+}\left[\sqrt{x}\cdot\sin\left(\frac{1}{x}\right)\right]$. [HINT: Use The Squeezing Theorem.]

5) Find $\lim_{x\to 0}\left[\frac{\sin(ax)}{x}\right]$ where a is a non-zero real number.

6) Find $\lim_{x\to +\infty}\left[\frac{\sin(x)}{x}\right]$, $\lim_{x\to +\infty}\left[x\sin(x)\right]$ and $\lim_{x\to +\infty}\left[x-\sin(x)\right]$.

7) Verify that $\lim_{x\to 0}\left[\frac{1-\cos(x)}{x^2}\right]=\frac{1}{2}$ using the following identity:

$$\frac{1-\cos(x)}{x^2}=\frac{1-\cos(x)}{x^2}\cdot\frac{1+\cos(x)}{1+\cos(x)}=\frac{1-\cos^2(x)}{x^2}\cdot\frac{1}{1+\cos(x)}$$

8) Use the limit in 7) to verify that $\lim_{x\to 0}\left[\frac{1-\cos(x)}{x}\right]=0$. [HINT: $\frac{1-\cos(x)}{x}=x\cdot\frac{1-\cos(x)}{x^2}$.]

Pledged Problem Set #2

HINT FOR 1) AND 2): Use your calculator to evaluate the left hand side of the given equation with y replaced by x^r or $x^s\ln(x)$. The *factor* function might be helpful as well.

1. Find all values of r such that the function $y=y(x)=x^r$, with domain $x > 0$, satisfies the equation:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0.$$

2. Consider the equation: $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$.

   (a) Find all values of r such that the function $y=y(x)=x^r$, with domain $x > 0$, satisfies the given equation.

   (b) Find all values of s such that the function $y=y(x)=x^s\ln(x)$, with domain $x > 0$, satisfies the given equation.

3. Suppose that $r > 0$. Let T be the tangent line to the graph of $y=\frac{1}{x^r}$ at the point $\left(a,\frac{1}{a^r}\right)$ where $a > 0$. Find the area of the triangle whose vertices are (i) the y-intercept of T, (ii) the x-intercept of T and (iii) the origin, and show that the area depends on the value of a for every r except $r = 1$. [HINT: The area depends of the value of r for every $r > 0$.]

4. Properties (I), (II) and (III) below are easily verified and you may use any of them, as desired, in working parts (a) and (b) below. [You DO NOT have to verify (I), (II) and (III).]

   (I) The sum of two even (respectively odd) functions is an even (respectively odd) function.

   (II) The product of two even (or two odd) functions is an even function.

   (III) The product of an even function and an odd function is an odd function.

(a) Show that (i) the derivative of an odd function, $f(x)$, is an even function and (ii) the derivative of an even function, $g(x)$, is an odd function. [HINT: In each case, start with the defining equation and differentiate using the Chain Rule.]

(b) Let $h(x)$ be an even function and consider the odd function $H(x) = x^3 h(x)$. [See (III) above.] Use (ii) from part (a) and as many of the properties (I), (II) and (III) as necessary to show directly that $H'(x)$ is an even function, thereby verifying (i) from part (a) in this special case. [HINT: Use the Product Rule to express $H'(x)$ in terms of $h(x)$ and $h'(x)$ and appropriate powers of *x*.]

Pledged Problem Set #3

1. [20 points] Find the values of *a*, *b*, *c* and *d* such that $ax^3 + bx^2 + cx + d$ has a relative minimum at the point (3, –15) and a relative maximum at the point (–1, 49). [HINT: You can use your calculator to solve the system of equations in *a*, *b*, *c* and *d* that corresponds to the four given conditions.]

2. [30 points] The results below can be used to approximate values of natural logarithms. You should use your calculator to verify the inequalities in parts (a) and (b) for at least one value of *x*, perhaps *x* = 0.06.

a) Let $f_1(x) = \ln(1+x) - \left(x - \frac{x^2}{2}\right)$ for $x > -1$. Use the Mean Value Theorem on the interval $[0, x]$ to show that if $x > 0$, then there is *c* such that

$0 < c < x$ and $\ln(1+x) - \left(x - \frac{x^2}{2}\right) = \frac{c^2 x}{1+c}$. Use this equation to conclude that

$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + x^3$ if $x > 0$.

b) Let $f_2(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \ln(1+x)$ for $x > -1$. As in part (a), show that if $x > 0$, there is *c* such that $0 < c < x$ and $f_2(x) = \frac{c^3 x}{1+c}$. Conclude that

$x - \frac{x^2}{2} + \frac{x^3}{3} - x^4 < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

3. [20 points] Let $f(x) = \frac{x}{x^2 + bx + c}$ for $x \geq 0$ where *b* and *c* are positive constants.

a) Show that the graph of *f* has an absolute maximum but no absolute minimum on the interval $(0, +\infty)$. [Note that 0 is NOT included in the interval.]

b) Find <u>all</u> values of *b* and *c* such the graph of *f* has a relative extremum at *x* = 2 and a point of inflection at *x* = 6.

4. [30 points] Suppose that *f* is a function whose <u>derivative</u> is given by $f'(x)=\begin{cases}\dfrac{\sin(2x)}{x} & \text{if } x\neq 0\\ 2 & \text{if } x=0\end{cases}$.

a) Explain why it is NOT possible to determine the value of *f(x)* for any value of *x*.

b) Find the value of $f''(0)$. [HINT: You will need to evaluate a limit to find this value.]

c) Find the *x* coordinate of each relative maximum and each relative minimum of *f* on the interval $\left(0,\dfrac{3\pi}{2}\right)$. [HINT: You WILL NOT need your calculator to find your answers to this part.]

d) Find the *x* coordinate of each point of inflection of *f* on the interval $\left(0,\dfrac{3\pi}{2}\right)$. Your answer(s) should be accurate to at least three places to the right of decimal point. [HINT: You WILL need your calculator to find your answers to this part.]

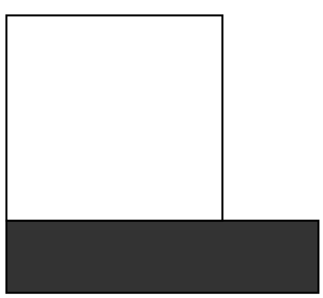
Pledged Problem Set #4

1. [30 points] A farmer wishes to fence off a rectangular enclosure using the long side of his barn (the black rectangle in the drawings below) as (possibly only part of) one side of the enclosure. The farmer has 250 feet of fencing and will use all of it. The resulting enclosure will resemble one of the two configurations below. The length of the long side of the barn is 100 feet. [The pictures below are not drawn to scale.]

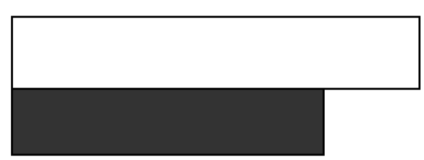
a) Express the area of the enclosure as single function of one independent variable. Be sure to specify the domain of your function. [HINTS: (i) Your answer will probably be a piecewise-defined function. (ii) You will probably want to use either the height or the width of the enclosure as the independent variable in your function.]

b) Find the absolute maximum of the function from part a) and use your answer to determine the dimensions of the rectangular enclosure of largest possible area.

One side of the enclosure is "all barn."



One side of the enclosure is part barn and part fencing.



2. [30 points] Let $f(x)=e^{-2x}$ for $x\geq 0$. For $a>0$, let *T* be the line that is tangent to the graph of *f* at the point $(a, f(a))$. Let *B* be the *y*-intercept of *T*, and let *C* be the *x*-intercept of *T*. Let $A=A(a)$ be the area of the triangle whose vertices are *B*, *C* and (0, 0). Find, with justification, the absolute maximum and the absolute minimum values of *A*. [HINT: It is possible that *A* has no absolute maximum and/or no

absolute minimum. If you decide that one or both of these possibilities is the case, be sure to justify your decision.]

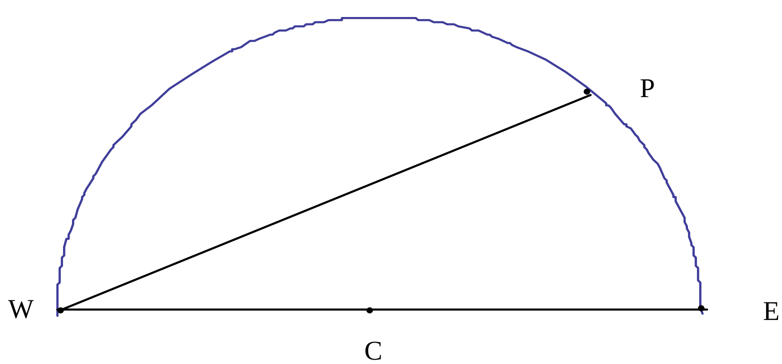
3. [40 points] A biathlete trains at a lake that is a perfect circle with radius 1000 feet. The northern half of the lake is represented in the figure below. Points *E* and *W* are the endpoints of a diameter, and *C* is the midpoint of this diameter; so *C* is the center of the circle. The biathlete begins at point *E* and runs along the shore of the lake counterclockwise until she reaches point *P*. Then the biathlete enters the water and swims directly to point *W*. The biathlete runs at a constant rate and swims at a constant rate that is exactly half of the rate at which she runs. [The biathlete is a much better swimmer than runner.]

a) Let θ be the measure of the central angle $\sphericalangle ECP$. Show that the length of segment *PW* is $2000\cos\left(\frac{\theta}{2}\right)$. [HINT: This is a geometry, or possibly a trigonometry problem; no calculus is required.]

b) Express the total time required to run from *E* to *P* and then swim from *P* to *W* as a function of the variable θ. Be sure to specify the domain of your function. [HINT: Introduce a parameter, perhaps *s*, to denote the rate at which the biathlete swims.]

c) Find the values of θ that give the absolute maximum and the absolute minimum for your function from part b).

d) At what rate, in feet per minute, does the biathlete swim if the absolute maximum value of the function from part b) is nine minutes?



Pledged Problem Set #5

1. Do each of the three parts of this problem.

a) Let $F(x)=\int_0^{\pi x}\frac{\sin(t/x)}{t}\,dt$. You can use your TI-89 to verify that $F'(x)=0$ for all $x > 0$. (i) Use the substitution $u=t/x$ to transform the definite integral in the dummy variable *t* to an integral in the dummy variable *u*. (ii) Use the transformed integral to explain why the derivative of *F* is 0.

b) Let $G(x)=\int_0^{\pi x}\sin(t/x)\,dt$. You can use your TI-89 to verify that $G'(x)=2$ for all $x > 0$. (i) Use the substitution $u=t/x$ to transform the definite integral in the dummy variable *t* to an integral in the dummy variable *u* multiplied by a function of *x*. (ii) Use the transformed integral to explain why the derivative of *G* is 2.

c) Let $H(x)=\int_0^{\pi x} t\sin(t/x)\,dt$. (i) Use your TI-89 to find the derivative of $H(x)$. (ii) Use the substitution $u=t/x$ to transform the definite integral in the dummy variable *t* to an integral in the dummy variable *u* multiplied by a function of *x*. (iii) Use your answers to (i) and (ii) to determine the value of the definite integral $\int_0^{\pi} u\sin(u)\,du$. [To earn full credit for this part, you MUST use your answers to (i) and (ii) to determine the value of the integral in (iii).]

2. This problem will require the use of Part 2 of the Fundamental Theorem of Calculus.

a. Suppose that *f* is a continuous function and that *p* is a real number. Find the Newton's Method Formula for the equation in the "unknown" *x*: $\int_0^x f(t)\,dt = p$. Your answer should include a definite integral.

b. Let $F(x)=\int_0^x \frac{\sin(t^2/3)}{t}\,dt$. The equation $F(x)=27/32$ has two exactly two positive solutions, one between 2 and 3, and the other between 3 and 4. Use your formula from part a), implemented on your TI-89, to find these two solutions, accurate to **at least eight places** to the right of the decimal point. [*You must enter the Newton's method formula into your calculator correctly so be careful with that step. Then you will need to be patient as your calculator does the computations to take you from one approximate solution to the next.*]

c) The function *F* from part b) must (and does) have a relative extremum at a value of *x* that falls between the two solutions from part b). (i) Find the *x* and *y* coordinates of this extremum, each exact or correct to **at least five places** to the right of the decimal point. (ii) Use the Second Derivative Test to determine whether the extremum is a relative maximum or a relative minimum.

3. The velocity of a particle moving on the *x*-axis is given by $v(t)=t^2\sin(t^2)$ for $0\le t\le 2.5$.

[NOTE: All numerical values given as answers in parts a) through d) should be exact or correct to at least four places to the right of the decimal point.]

a) Find all intervals on which the particle is speeding up and all intervals on which the particle is slowing down. Be sure to justify your answers.

b) If the particle is at $x = 6.7$ when $t = 1$, determine the position of the particle at $t = 2.5$ and at $t = 0$.

c) Find the total distance traveled by the particle as *t* goes from 0 to 2.5.

d) Find the right-most and the left-most positions of the particle on the *x*-axis as *t* goes from 0 to 2.5. I.e. find the maximum and the minimum of *x(t)* on the interval [0, 2.5].

Density Functions
Lin Mc Mullin - National Math and Science Initiative

When you look at the just released 2008 AB Calculus exam you will find a "new" topic: density (question 92). This topic has been hinted at for the last few years with a similar question in the Course Description booklet. Both of these questions will be discussed below. My advice is not to make too big a deal of this, but if you have time you can take a look. Should this kind of question appear on the free-response section I would guess that the question will be carefully worded so that students who never saw this kind of question would have a good chance of responding with a correct answer.

The Mathematics

A *density function* gives the amount of something per unit of length, area or volume. Examples:

- The density of a metal rod may be given in units of grams/centimeter.
- The density of the population of a city may be given in units of people/square mile.
- The density of a container of substance may be given in pounds/cubic foot.

The density can be used to find the amount. In each example, notice that the length, area or volume of the region is multiplied by the density to find the amount.
Examples:

- A 10 cm rod with a constant density of 3 g/cm has a mass of
  $10 \text{ cm} \cdot \frac{3 \text{ grams}}{\text{cm}} = 30 \text{ grams}$
- In other situations the density is not constant and is given by some function. Suppose our metal rod of length *b* has a density of $\rho(x)$ grams/cm where *x* is measured from one end of the rod. To find the total mass we think of cutting the rod in the very small pieces. (Think partition; each piece has a length of Δx in which the density is nearly constant say $\rho(x_i)$.) The sum of the mass of these pieces is the Riemann sum $\sum_{i=1}^{n} \rho(x_i)\Delta x$. The limit of this expression as $\Delta x \to 0$ gives the total mass in grams:
  $$M = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \rho(x_i)\Delta x = \int_0^b \rho(x)\,dx$$
  Notice that $\sum_{i=1}^{n} \Delta x$ is the length of the rod. This is multiplied by the density to find the mass.
- The next example is from the Course Description book.
  A city is built around a circular lake that has a radius of 1 mile. The population density of the city is $f(r)$ people per square mile, where *r* is the distance from the center of the lake, in miles. Which of the following expressions gives the number of people who live within 1 mile of the lake?

  (A) $2\pi\int_0^1 r\,f(r)\,dr$

  (B) $2\pi\int_0^1 r\big(1+f(r)\big)\,dr$

  (C) $2\pi\int_0^2 r\big(1+f(r)\big)\,dr$

  (D) $2\pi\int_1^2 r\,f(r)\,dr$

  (E) $2\pi\int_1^2 r\big(1+f(r)\big)\,dr$

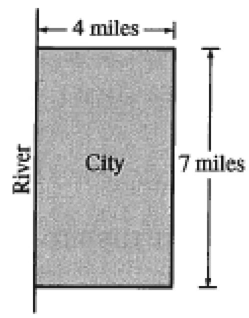
We need to partition the region so that each piece has a close to a constant density. Thin rings around the lake will accomplish this. A ring, if straightened out, is similar to a rectangle of length $2\pi r_i$ where r_i is the distance from the center of the lake (this is the circumference of the ring), the width of this ring (rectangle) is Δr. In this ring (rectangle) the population density is $f(r_i)$ people per square mile, so the population in the ring is approximated by multiplying the *area* by the *density*: $2\pi r_i f(r_i)\Delta r$. Adding these gives a Riemann sum whose limit gives the total population:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi r_i f(r_i)\Delta r = 2\pi \int_1^2 r f(r)\, dr$$

Answer (D). The limits of integration are from the edge of the lake, $r = 1$ to $r = 2$ ("one mile from the lake"). Another way to look at this is that $\sum_{i=1}^{n} 2\pi r_i \Delta r$ is the area of the city; this is multiplied by the population density to find the population.

This idea is called in some textbooks a *radial density function*. The fact that the answer looks like the formula for volume by cylindrical shells is not quite an accident.

- 2008 AB Calculus exam #92:



A city located beside a river has a rectangular boundary as shown in the figure above. The population density of the city at any point along a strip *x* miles from the river's edge is $f(x)$ people per square mile. Which of the following expressions gives the population of the city?

(A) $\int_0^4 f(x)\,dx$

(B) $7\int_0^4 f(x)\,dx$

(C) $28\int_0^4 f(x)\,dx$

(D) $\int_0^7 f(x)\,dx$

(E) $4\int_0^7 f(x)\,dx$

A thin vertical strip of the city x_i miles to the right of the river has an area of $7\Delta x$. The population in each such strip is found by *multiplying the area by the density function*; this gives $7 f\left(x_i\right)\Delta x$. These are then added forming a Riemann sum, etc.

$$\text{Population} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 7 f\left(x_i\right)\Delta x = 7\int_0^4 f\left(x\right)dx$$

Answer (B)

Alternative solution: When I first saw this question, not having thought about density for quite a while, I answered it by doing a unit analysis. Since unit analysis is a good thing for students to understand I'll outline my thinking next.

We are looking for the population so the answer must be in units of "people." The density function is in units of "people/square mile" (given). Both *x* and *dx* are in units of "miles" and the "7" also has units of "miles." Therefore, the only choice that gives people is the one that multiplies the 7, the *dx* and the density function. This is eliminates (A) and (D). The 28 in (C) must be square miles, making the overall units "people-miles" which is not what we're going for. Finally, choice (E) is eliminated since the 7 and the *dx* are not in the same direction. This leaves (B).

- A volume problem adapted from *Calculus* by Hughes-Hallett, Gleason, *et al:*

  The density of air *h* meters above the earth's surface is $\rho\left(h\right) = 1.25e^{-0.00012h}$ kg/m^3. Find the mass of a column of air 25 km high with a square base 3 meters on a side sitting on the surface of the earth.

  At any height, h_i meters above the earth the volume of a thin slice of the column of air is $3^2\Delta h$. The mass of this slice is $3^2\rho\left(h_i\right)\Delta h$. The sum of these slices gives a Riemann sum whose limit gives the total volume:

$$M = \lim_{\Delta h \to 0} \sum_{i=1}^{n} 9\rho\left(h_i\right)\Delta h = 9\int_0^{25,000} 1.25e^{-0.00012h}dh \approx 93,750 \text{ kg.}$$

References and Problem Sources

- The best textbook for this is *Calculus* by Hughes-Hallett, Gleason, *et al.* Section 8.4. (Omit the Center of Mass discussion.) There are a number of accessible problems in the exercises for this section.
- Next best is probably *Calculus* by Rogawski. Section 6.2 Exercises 24 – 32.
- Foerster (2nd edition) mentions only volume problems and has some good exercises (Section 11-3)
- Stewart's *Calculus Concepts and Contexts* (2nd edition) mentions it in three examples. Check the index in other editions of Stewart.
- Finney (FDWK) has a single population density problem (Section 7.1 #23) as part of a good discussion of modeling net change.
- Larson and Anton mention the topic very briefly with no good examples.

Upcoming Conferences

T^3 Conference
(Teachers Teaching with Technology)
March 6-7, 2010 Atlanta, Georgia
Special *conference within a conference* on Issues, Ideas, Innovations for technology in teaching calculus. This should be of special interest to AP Calculus teachers. The T^3 conference is March 5-7 and the conference within a conference would be at most Saturday/half day Sunday on March 6-7.

NCTM
San Diego
April 21-24, 2010
www.nctm.org

NCCTM
Greensboro, NC
October 28-29, 2010

NCTM Regional Conferences
Oct 7-8 Denver, CO
Oct 14-15 Baltimore, MD
Oct 28-29 New Orleans, LA

The **SCCTM/T^3** Regional conference will be held October 21-22, 2010 in Greenville, SC at the Carolina First Center. Speaker Proposal forms are located on the SCCTM web site www.scctm.org. For additional information contact Sherri Abel - sherriabel@charter.net

Summer Institutes (please check the College Board Website for more)
UNC-Charlotte June 28-July 2
UNC-Asheville June 28-July 1

Sixth U.S. Conference on CAS in Secondary Mathematics

Computer algebra systems (CAS) have the potential to revolutionize mathematics education at the secondary level. They do for algebra & calculus what calculators do for arithmetic: simplifying expressions, solving equations, factoring, taking derivatives, and much more!

Discover how secondary and middle school teachers are using CAS in their own classrooms. Get classroom tested ideas developed for CAS-enhanced classrooms. Learn what other countries are doing with CAS. Interact with prominent CAS pioneers from the USA and beyond.

WHEN:	Saturday, June 26, 2010 8:15 AM - 4:15 PM Sunday, June 27, 2010 8:00 AM - 1:00 PM
WHERE:	New Trier High School (Northfield Campus), 7 Happ Road, Northfield, IL 60093
COST:	Registration: **$195 (before May 7, 2010)** $250 (on or after May 7, 2010) *(Fee includes continental breakfast, box lunch, snack, and conference shirt)* **Optional Saturday evening social event**
HOTEL:	Renaissance Chicago North Shore Hotel, Northbrook IL. $99/night conference rate available until June 3, 2010 *(rate available for reservations June 24 to June 27, 2010)* Book directly online at http://cwp.marriott.com/chinb/meecas/ 1-800-468-3571 Mention group code "USACAS" **or** MEECAS"
HOW:	On-line registration, updates, and hotel information available beginning February 2010 at http://usacas.org

For more information , contact Ilene Hamilton at ihamilton@d125.org, Natalie Jakucyn at njakucyn@glenbrook.k12.il.us, or Pat Bowler-Johnson at bowlerjp@newtrier.k12.il.us
Sponsored in part by: *Texas Instruments, New Trier High School, Casio.* Organized by MEECAS (Mathematics Educators Exploring Computer Algebra Systems)

Items of Interest

Moodle and AP Calculus

Have you heard about Moodle? I wanted to start an on-line forum with my AP Calculus students for 2009-2010. I investigated several possibilities and then talked with my district technology department. We already had a server set up for Moodle so I decided to give it a try. It has been a wonderful experience. We have had several forum discussions and on-line chats on the evening before a test. Starting in January, I began posting weekly writing assignments. On the forums, all students could read what others posted. With the assignments, the students don't get to see the postings of anyone else. The first writing assignment was about calculus and the movies. The writing prompt was: If AP Calculus was a movie, it would be ______. Name the actual movie. Give the year it was released and the major actor/actress. Write a short paragraph explaining why you picked this movie.

I got some great responses. With its current popularity, **Avatar** was frequently discussed. Here are few excerpts: If AP Calculus was a movie, it would be "Legally Blonde" (2001, Reese Witherspoon). When Elle (Reese Witherspoon) walked into Harvard, she felt very intimidated. It seems as though she might fail. This is how I feel sometimes when I walk into Calculus. But in the end, she conquers her feelings of failure and graduates from Harvard. Although I don't plan to go to Harvard to be a lawyer, I do plan to succeed in Calculus!

If AP Calculus were a movie, it would be "Mission Impossible 3" (2006, Tom Cruise). Just like Ethan Hunt, we started with what seemed to be an almost impossible mission: to make it through the class and pass the AP exam. Although we have faced some of our toughest enemies (aka derivatives, limits, anti-derivatives, definite integrals, etc. etc.), lost some things very important to us (aka not done so well on tests and quizzes), and have been completely clueless and out of it at times, if we continue to work hard at it and not give up, hopefully it will be a success. And just like Ethan Hunt, we will make an impossible mission a **POSSIBLE** mission. "Expect the impossible..."

If AP Calculus was a movie, it would be "Avatar" (2009, Sam Worthington). In the movie the main character finds himself in a new, unknown world where he must learn the ways of the "Navi" race in order to succeed and be accepted in their world. Many times throughout the movie he gets discouraged and it seems as if he will fail, but in the end he makes it due to perseverance and is victorious. Calculus is similar to this. We, the students, are like the main character in the movie. We entered into this class, our new unknown world. Now we must slowly learn the ways of Calculus so that we can succeed and make a good grade in the class. We may be discouraged by new concepts and things we don't understand, but if we push through and keep working, in the end we will also be victorious.

And my favorite: "The Ugly Truth" (2009, Katherine Heigl, Gerard Butler) I chose The Ugly Truth because this is a movie about relationships and what girls want out of the relationship and what a guy wants out of it... Well I can compare this to calculus because first of all girls are very difficult to understand and so is calculus. Guys never know why girls think the way they do and I never know why calculus is the way it is. Also, most of the guys in our class just want the good grade in calculus while most girls are in it for the knowledge. This is how calculus is like the Ugly Truth.
Vicki M. Carter, T^3 National Instructor, Math Department Chairman, West Florence High School, Florence, SC 29501, 843-664-8472 carterv@bellsouth.net

New Technology

The TI-Nspire Navigator system was released in December 2009. This new Navigator system is completely wireless. Teachers can send documents wirelessly to the students' handhelds. There are also tools to collect, grade, save and record students' work. Quick polls and screen captures may be saved in a portfolio. One of the best features is the ability to select a presenter. Any student can present their calculator screen from anywhere in the room. For additional information, visit the Texas Instruments web site - http://education.ti.com/educationportal/sites/US/productDetail/us_nspire_navigator.html

Theorem of the Day

Consider using a new theorem every day in your classrooms. Go to http://www.theoremoftheday.org/ upper right corner and specifically theorem # 165 http://tinyurl.com/LinMc
Lin Mc Mullin, Director of Mathematics Programs, National Math and Science Initiative

Exam Review Ideas

Why wait until right before the AP exam to begin reviewing? At this time of year, the students have learned enough Calculus to begin tackling some AP problems. In addition to regular homework assignments, I assign a Problem of the Week (POTW). Each Friday the students receive an actual previous year's AP problem as the POTW. They either get a paper copy or a link to a file if I am feeling more "green." The first week they also receive the Free Response instructions right off AP Central. The problem is due the following Friday with complete work shown. It is worth the standard 9 points and is graded using the AP standard. Since it is review and intended to be a learning experience, during the week students may use their books or discuss the problems with each other or with me. They MAY NOT access rubrics through AP Central. It takes them a little while to realize these are actual AP problems and have solutions on AP Central, but eventually they figure it out. To keep them honest, about the second and fourth weeks I use either a secure Free Response provided through the AP audit site, an AP problem from before 1998, or I modify an old AP problem myself. Each Monday they get their problems back graded AP style and we spend a few minutes discussing results, problem emphasis, common errors and grading philosophies. This process has greatly increased the students' comfort level with the free response potion of the exam.
Greg Hill Hinsdale Central High School, Hinsdale Illinois

Explanations for Understanding Confusing Topics

Maria Ferraro (steckferr@tampabay.rr.com), retired from The Miami Valley School, Dayton, Ohio shared that she has used the explanations below in her classroom. She knows they were not original but has lost the source. If anyone knows the author(s), we will be glad to share this in our next newsletter.

Pumping Liquid: Think of a glass of water with a straw in it. The water is what you are pumping, and the straw is the apparatus through which the water is pumped. Ignore the water in the glass, and instead, notice where the water level is in the straw. It is precisely at the water level in the glass. The straw only needs to pull the water from the level of the water in the glass to the top of the glass. You aren't pumping from the bottom of the glass, because gravity acting on all the water outside the straw has already pushed whatever water may have initially been at the bottom of the straw up to the surface level of the water in the glass. As you continue to pump water, gravity continues to help by pushing more water up to the new surface level as the straw removes all the water. We only have to lift the water from the surface level of the liquid.

Velocity/Acceleration: If velocity and acceleration are in the same direction, then the speed is increasing, and if they are in opposite directions, the speed is decreasing. Another way to look at it, if the acceleration and velocity have the same sign, the speed is increasing, if they have opposite signs, the speed is decreasing. Consider throwing a ball upwards. It will slow down, reach a maximum height, and then speed up on its way back down. When it is traveling up, it will be slowing down. The velocity is positive due to its direction and acceleration is negative. When it heads back down, its velocity due to direction is negative and acceleration is also negative, so the speed will be increasing.

Presidential Award Information

Please consider applying yourself or nominating someone for the Presidential Awards for Excellence in Mathematics and Science Teaching (PAEMST). The following resources and media are available for your use: The PAEMST website http://paemst.org or specifically, http://recognition.paemst.org/ includes information about this year’s winners and their trip to DC.

* Professional headshots and award ceremony photos, available for download.
* Links to White House press releases and the "Educate to Innovate" video/webcast, featuring PAEMST awardees.
* Link to the National Science Foundation press release, which can also be found at

http://nsf.gov/news/news_summ.jsp?cntn_id=116188&org=NSF&from=news

The White House Blog, featuring Barbara Stoflet, http://www.whitehouse.gov/blog/2010/01/06/educate-innovate-awarding-excellence-math-teaching

Online Resources for Teachers: http://www.teachersdomain.org